A Theoretical Approach of the Electrical Conductance in Nanostructures

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Using the scale relativity theory it is shown that the “mechanical” behaviour of the nanostructures in an arbitrary fractal dimension has a hysteretic character by means of a generalized Navier-Stokes type equation with an imaginary viscosity coefficient. At microscopic scale, i.e. for the fractal dimension, $D_F = 2$, *the nanostructure behavior is described by a generalized Schrödinger equation. In the hydrodynamic formulation of the same theory, the quantization of the electrical conductance and particularly the Landauer’s result are obtained. A mechanism of electronic transport through quantum rings is proposed.*

Keywords: nanostructures, fractal dimension, scale relativity theory, conductance

The paper approaches a new thematic, developing a robust physico-mathematical model due to some previous results [1], but the focus being this time on the electrical conductance calculus in the nanometer-sized structures.

The existence of a minimum value of the electrical conductance (electrical conductance quanta), $G_0 = e^2/h$ with e the electron charge and *h* the Planck constant, is a well-established fact in the physics of mesoscopic structures [2-4]. The minimum conductance value is derived from an uncertainty principle and also from quantum transport considerations in an ideal one-dimensional ballistic conductor subjected to an external electric potential that induces different quasi-Fermi-energies in the reservoirs between which the conductor is sandwiched [2,3]. Numerous experiments have validated this theoretical result and demonstrated that the conductance varies in steps (for details see references [2-4]).

On the other hand, recent results on the transport phenomena at nanometer scale [5-7] and turbulence phenomena in nanoscale conductors [8-13] require the development of new “scale” physical theories, *i.e.* of fractal space-time type (for example the scale relativity (SR) model and transfinite physics [14,15], in which the macroscopic scale specific to the classical quantities coexist and it is compatible, simultaneously, with the microscopic “scale” specific to the quantum quantities. In this sense, a realistic model should take into account three major considerations. Firstly, the semi-quantum physical theories must not be imposed, but can be generated as transitions between the interaction scales. Secondly, the topological dimension and implicitly, the fractal one (for details see [16,17], induces new transport mechanisms. Thirdly, the so-called anomalies, *e.g.* the increases of the electrical and thermal conductivity in nanostructures, appear as natural phenomenon in the context of material structures self-organization by means of the spontaneous symmetry breaking (for details see [8-13] and [18]).

In the present paper a theoretical approach of the electrical conductance of electrons in nanostructures using the SR is established. The paper is structured in the standard sections, as follows. In Theoretical Part section, the mathematical model is elaborated, having as final result the hydrodynamic formulation. In Results and Discussions section, the electrical conductance quanta and its quantification are obtained.

Experimental part

The theoretical description of microphysical systems is generally based on Schrödinger’s wave mechanics, Heisenberg’s matrix mechanics, or on Feynman’s path-integral mechanics. Another approach is the hydrodynamic formulation of quantum mechanics. SR [14,15] is a new approach to understand quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems. It is based on a generalization of Einstein’s principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as speed characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of SR). The principle of SR is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution.

According to SR [14,15], a non-differentiable continuum is necessarily fractal and the trajectories in such a space (or space-time) own (at least) the following three properties:

i) the test particle can follow an infinity of potential trajectories: this leads to the use a fluid-like description;

ii) the geometry of each trajectory is fractal (of fractal dimension D_F [16,17] - or particularly of fractal dimension $D_F = 2$ as in the Nottale’s approach of the SR. Each elementary displacement is then described in terms of the sum, $dX = dx + d\zeta$, of a mean classical displacement $d_x = vdt$ and of a fractal fluctuation $d\zeta$,whose behaviour

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satisfies the principle of SR (in its simplest Galilean version). It is such that $\langle d\zeta\rangle = 0$ and $\langle d\zeta\rangle = 2D(d\tau)^{(2/DF)-1}$where D defines the fractal/non-fractal transition, *i.e.* the transition from the explicit scale dependence to scale independence. The existence of this fluctuation implies introducing new second order terms in the differential equation of motion;

iii) time reversibility is broken at the infinitesimal level: this can be described in terms of a two-valuedness of the velocity vector for which we use a complex representation, $V=(v_+ + v_-)/2 - i(v_+ - v_-)/2$. We denoted by v_+ the "forward" speed and by v_- the "backward" speed.

These three effects can be combined to construct a complex time-derivative operator [14,15]

$$\frac{\delta}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla - iD(dt)^{(2/D_F)-1}\Delta \qquad (1)$$

so that, the first Newton's principle in its covariant form becomes $\delta V / dt = 0$, i.e.

$$\frac{\delta \boldsymbol{V}}{dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - \boldsymbol{V}\times(\nabla\times\boldsymbol{V}) - iD(dt)^{(2/D_F)-1}\Delta\boldsymbol{V} = 0 \qquad (2)$$

The equation (2) is a generalized Navier-Stokes type equation with an imaginary viscosity coefficient $\eta = iD(dt)^{(2/DF)-1}$. Then, the "mechanical" behaviour of the nanostructures is viscoelastic type or hysteretic type [19, 20]. Such a result is in agreement with the opinions given in [1,2] and [20]. The nanostructures can be described at the macroscopic scale by Kelvin-Voight or Maxwell rheological model with complex structure coefficients.

If we suppose that the motion of the microparticles is irrotational, $\nabla \times V = 0$, we can choose V of the form:

$$\boldsymbol{V} = -2iD(dt)^{(2/D_F)-1}\nabla(\ln\psi) \qquad (3)$$

Then equation (2) takes the form

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - iD(dt)^{(2/D_F)-1}\Delta\boldsymbol{V} = 0 \qquad (4)$$

and ψ satisfies a Schrödinger type equation:

$$D^2(dt)^{(4/D_F)-2}\Delta\psi + iD(dt)^{(2/D_F)-1}\frac{\partial\psi}{\partial t} = 0 \qquad (5)$$

For $D_F = 2$ and $2mD = h$, with η the Plank's reduced constant and m the rest mass of the particle test, the standard Schrödinger equation results. In such conjecture, our model describes the behaviour of nanostructures at the microscopic scale.

Let us consider the wave function, $\psi = \sqrt{\rho}e^{iS}$, with $\sqrt{\rho}$ the amplitude and S the phase of ψ. Then, the complex speed field (4) in the form

$$\boldsymbol{V} = \boldsymbol{v} + i\boldsymbol{u} \qquad (6a)$$

has the components:

$$\boldsymbol{v} = 2D(dt)^{(2/D_F)-1}\nabla S, \qquad \boldsymbol{u} = -D(dt)^{(2/D_F)-1}\nabla\ln\rho \qquad (6b,c)$$

Introducing (6a-c) in (4) and separating the real and imaginary parts, we obtain:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{\boldsymbol{v}^2 - \boldsymbol{u}^2}{2} - D(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u}\right) = 0$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\boldsymbol{v}\cdot\boldsymbol{u} + D(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{v}\right) = 0 \qquad (7a,b)$$

or, up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ

$$\left(m\frac{\partial \boldsymbol{v}}{\partial t} + m(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla(Q)$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \qquad (8a,b)$$

with Q the fractal potential:

$$Q = -2mD^2(dt)^{(4/D_F)-2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{m\boldsymbol{u}^2}{2} - mD(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u} \qquad (9)$$

Naturally, several preliminary conclusions can now be expressed.

Any particle is in a permanent interaction with a "subquantum level" through the fractal potential Q.

The "subquantum level" is identified with a non-relativistic quantum fluid described by the hydrodynamic equations (8a,b).

The fractal potential depends only on the imaginary part u (it is called the *osmotic speed* [14,15] and [21]) of the complex speed field V and comes from the non-differentiability of the fractal space-time.

The compatibility between the non-relativistic hydrodynamic model and the wave mechanics (the scale compatibility of interaction) [21] implies, through the relation (6b), the quantization condition

$$\oint m\boldsymbol{v}d\boldsymbol{r} = 2mD(dt)^{(2/D_F)-1}\oint dS = 4\pi nmD(dt)^{(2/D_F)-1}, \quad n = 1, 2, ... \qquad (10)$$

For $D_F = 2$ and $D = \eta / 2m$ the relation (10) takes the standard form $\oint m\boldsymbol{v}d\boldsymbol{r} = nh$.

In the ground states, *i.e.* for the quantum numbers $n = 1$, $l = m = 0$, the state density [22] is

$$\rho(r) = \frac{1}{\pi r_0^3}e^{-\frac{2r}{r_0}}. \qquad (11)$$

Substituting this result in the fractal potential expression (9), *i.e.*

$$Q(r) = -\frac{2mD^2(dt)^{(4/D_F)-2}}{\sqrt{\rho}}\left(\frac{d^2\sqrt{\rho}}{dr^2} + \frac{2}{r}\frac{d\sqrt{\rho}}{dr}\right) =$$
$$= -\frac{2mD^2(dt)^{(4/D_F)-2}}{r_0}\left(\frac{1}{r_0} - \frac{2}{r}\right), \qquad (12)$$

we obtain the attractive force,

$$F(r) = -\frac{\partial Q}{\partial r} = -\frac{4mD^2(dt)^{(4/D_F)-2}}{r_0}\frac{1}{r^2} \qquad (13)$$

Consequently, the 'subquantum level' becomes a force field source of an attractive type. Now, at the microscopic scale, *i.e.* for $D_F = 2$ and $2mD = \eta$, if $r_0 = \eta / \alpha mc$, where α is the fine-structure constant, the previous relation becomes the electrostatic force $F(r) = e^2 / 4\pi\varepsilon_0 r^2$, a result according to the hysteretic behaviour of the nanostructures;

The field $\rho(r, t)$ is a probability distribution, namely the probability of finding the particle in the vicinity dr of the point r at time t,

$$dP = \rho d\boldsymbol{r}, \quad \iiint \rho d\boldsymbol{r} = 1, \qquad (14a,b)$$

the space integral being extended over the entire area of the system. Any time variation of the probability density $\rho(r, t)$ is accompanied by a probability current ρv pointing towards or outwards, the corresponding field point r (8b). Therefore, the equation (8b), by means of equation (14a, b), corresponds to the Born's postulate.

The position probability of the real speed field v(r, t) (equation (8a)), varies with space and time similar to a hydrodynamic fluid placed in a fractal potential. The fluid (in the sense of a statistical particles ensemble) exhibits, however, an essential difference compared to an ordinary fluid: in a rotation motion $v(\boldsymbol{r}, \boldsymbol{t})$ increases (decreases) with the decreasing (increasing) distance $\boldsymbol{r}$ from the center (10).

The expectation values for the real speed field and the speed operator [14,15], $\hat{v} = -2iD(dt)^{(2/D_F)-1}\nabla$,, of the model are equal,

$$\langle v\rangle = \iiint \rho v d\boldsymbol{r} = \iiint \Psi^* \hat{v}\, \Psi d\boldsymbol{r} = \langle \hat{v}\rangle_{Wave-Mechanics} \qquad (15)$$

but in the higher-order, $|n| > 2$, similar identities are invalid, namely $\langle v^n\rangle \neq \langle v^n\rangle_{Wave-Mechanics}$. The expectation for the 'fractal force' vanishes at all times (theorem of Ehrenfest [21,22]), *i.e.*,

$$\langle -\nabla Q\rangle = \iiint \rho(-\nabla Q)d\boldsymbol{r} = 0 \qquad (16)$$

or explicitly

$$2mD^2(dt)^{(4/D_F)-2}\iiint \rho\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)d\boldsymbol{r} =$$
$$= mD^2(dt)^{(4/D_F)-2}\oint(\rho\nabla\nabla\ln\rho)\cdot d\boldsymbol{\sigma} = 0 \qquad (17)$$

Two types of stationary states are to be distinguished:

1) *Dynamic states*. For $\partial/\partial t = 0$ and $v \neq 0$, equations (8a,b) give

$$\nabla\left(\frac{1}{2}mv^2 - \frac{m\boldsymbol{u}^2}{2} - mD(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u}\right) = 0,\quad \nabla(\rho\boldsymbol{v}) = 0 \quad (18a, b)$$

namely,

$$\frac{1}{2}mv^2 - \frac{m\boldsymbol{u}^2}{2} - mD(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u} = E, \quad \rho\boldsymbol{v} = \nabla\times\boldsymbol{F} \quad (19a,b)$$

Consequently, inertia, *mv* . ∇*v* and fractal force, -∇*Q*, are in balance at every field point (equation (18a)). The sum of the kinetic energy, $mv^2/2$ and fractal potential, *Q*, is invariant, *i.e.* equal to the integration constant E ≠ E(r) (equation (19a)).

E ≡ <*E*> represents the total energy of the dynamic system. The probability flow density ρv has no sources (equation (17b)), *i.e.* its streamlines are close (equation (19b)).

2) *Static states*. For $\partial/\partial t = 0$ and $v = 0$, equations (8a, b) give

$$\nabla\left(-\frac{m\boldsymbol{u}^2}{2} - mD(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u} + U\right) = 0, \qquad (20)$$

i.e.

$$-\frac{m\boldsymbol{u}^2}{2} - mD(dt)^{(2/D_F)-1}\nabla\cdot\boldsymbol{u} + U = E \qquad (21)$$

The fractal force, -∇*Q*, at any field point has the same null value (equation (17)). The fractal potential, *Q*, is invariant, *i.e.* equal to the integration constant E ≠ E(r) (18). Equation (8b) is identically satisfied. *E* ≡ <*E*> represents the total energy of the static system.

Results and discussion

Electrical conductance quanta. Landauer's conductance

Let us show that at the microscopic scale, *i.e.* $D_F = 2$ and $2mD = \eta$, we can derive from equation (19a) the quantized conductance of an ideal quasi-1D liquid. We consider the electron liquid adiabatically connected to two reservoirs, and we call $v_{L(R)}$ and $\mu_{l(R)} = Q$ the speed and the chemical potential, respectively, in the left (right) reservoir, with $\mu_R - \mu_L = eV$ with *V* the potential. Then, the equation (18a) becomes:

$$\frac{v_L^2}{2} + \frac{\mu_L}{m} = \frac{v_R^2}{2} + \frac{\mu_R}{m} \qquad (22)$$

By denoting the flow velocity, $v = (v_L + v_R)/2$, and the co-moving Fermi velocity [24], $v = (v_L - v_R)/2$, we get from equation (22) the relation $2mvv_F = eV$. By definition, the current is given by $I = env$ so that, by using the 1D density of states and $I = emvv_F/\pi\eta = e^2V/h$, which, in the linear regime, gives the "quantized" conductance (per spin): $G_0 = I/V = e^2/h$. If we assume that only a fraction *T* of electrons is transmitted due to the presence of a barrier in the electron fluid, we can argue that, in linear response, the current is an equal fraction of the current in the absence of the barrier, *i.e.* $I = envT$. The conductance is thus $G = T \cdot e^2/h$ in accordance with Landauer's results [23,24].

The electric conductance quantification

If the principle of local equivalence works in a fractal space-time, the mechanical momentum equals the electromagnetic one, $p = mv = qA$ with *A* the vector potential and *q* the effective charge of a test particle. Then, the equation (10) in the form

$$\oint \boldsymbol{A}d\boldsymbol{r} = \iint_{\Sigma}\boldsymbol{B}d\boldsymbol{S} = \Phi_n = 4\pi n\frac{mD(dt)^{(2/D_F)-1}}{q}, \quad n = 1, 2, \ldots \qquad (23)$$

implies the quantization relation of the magnetic flux Φ_n of fractal type,

$$\Phi_n = n\Phi_0 \qquad (24)$$

In the relation (24) $\Phi_0 = 4\pi m D(dt)^{(2/D_F)-1}/q$ is the magnetic flux quanta of fractal type.

We name the electrical conductance of fractal type the expression

$$G_n = \frac{Nq}{\Phi_n}. \qquad (25)$$

Each flux line has *l* charge carriers attached on it, and

$$l = \frac{\Phi_n}{\Phi_0}, \qquad (26)$$

is the number of states per Landau level. So, the number *N* of the charge carriers (according to Pauli's principle) is

$$N = nl. \qquad (27)$$

Considering the relation (27), the relation (25) can be written as

$$G_l = l\frac{q^2}{4\pi m D(dt)^{(2/D_F)-1}}, \qquad (28)$$

At the microscopic level, *i.e.* $D_F = 2$, $2mD = \eta$ and $q = e$, the relation (28) takes the standard form

$$G_l = l\frac{e^2}{h} = lG_0 \qquad (29)$$

with G_0 the electrical conductance quanta.

Such result can be experimentally proved by measuring the electron transport through a lithographically defined quantum ring in the Coulomb blockade regime [25].

Conclusions

A theoretical model of the electrical conductance in nanostructures, using the scale relativity is established. By means of a scale covariance form of the Newton's equation, a generalized Navier-Stokes type equation with an imaginary viscosity coefficient is obtained. Then, the "mechanical" behaviour of the nanostructures can be of viscoelastic type or hysteretic type.

A Schrödinger type equation is deduced from an irrotational movement equation of the microparticles. A hydrodynamic model in the scale relativity context has been developed. In such conjecture the compatibility of interaction scales implies the quantification of the electrical conductance and the energy equation leads to the identification of the Landauer's conductance. All these results certify the hypothesis that the electronic transport in nanostructures takes place through quantum rings.

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